Localized Ideal and Resistive Instabilities in 3-dimensional MHD Equilibria with Closed Field Lines

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A class of stability criteria is derived for MHD equilibria *with closed field lines.* The destabilizing perturbations have finite gradients along the field and are localized around a field line, the localization being stronger on the pressure surface than in the radial direction. By contrast, in sheared configurations the localization is comparable in both directions. The derived stability criteria are less stringent than those obtained for MHD equilibria with shear for similarly localized perturbations in the limit of low shear. These results, obtained from the energy principle, are a particular case of those obtained by solving the linearized resistive MHD equations with an appropriate ansatz and subsequently taking the limit of vanishing shear and resistivity.

In previous papers [1–3], criteria for the stability of symmetric and also of nonsymmetric MHD equilibria with shear in respect of ideal and resistive localized perturbations were derived. For the sake of completeness, and owing to the interest given to shearless configurations, a class of stability criteria for localized perturbations is presented here. The derivation clearly shows the differences (and also the similarities) of the destabilizing perturbations in the cases with and without shear.

The perturbations considered here leave the plasma surface unchanged. For these, the change δW in potential energy is

$$\delta W = \frac{1}{2}\int\Bigg[\frac{|\boldsymbol{Q}\times\boldsymbol{B}|^2}{B^2} + \frac{|\boldsymbol{Q}\cdot\boldsymbol{B}-\boldsymbol{\xi}\cdot\nabla p|^2}{B^2} + \gamma_H\, p\,|\nabla\cdot\boldsymbol{\xi}|^2 + \frac{\boldsymbol{B}\cdot\boldsymbol{J}}{2\,B^2}\{(\boldsymbol{B}\times\boldsymbol{\xi}^*)\cdot\boldsymbol{Q}+(\boldsymbol{B}\times\boldsymbol{\xi})\cdot\boldsymbol{Q}^*\} - (\boldsymbol{\xi}\cdot\nabla p)(\boldsymbol{\xi}^*\cdot\boldsymbol{\varkappa}) + (\boldsymbol{\xi}^*\cdot\nabla p)(\boldsymbol{\xi}\cdot\boldsymbol{\varkappa})\Bigg]\,\mathrm{d}\tau, \quad (1)$$

where all the symbols have their usual meaning and are explained in detail elsewhere [1–3].

To describe the system, coordinates υ, ϑ, $\varphi=\zeta-q\,\vartheta$, are employed, where υ, ϑ, ζ are Hamada coordinates and $q=\dot\Psi/\dot\chi=M/N$ (M, N integers) is the safety factor, which is the same for all field lines. In these coordinates, the physical quantities are periodic in ϑ with period N, and in φ with period 1. At constant φ and υ, ϑ measures the angle along a particular field line, which closes upon itself after a poloidal angle of $\Delta\vartheta=N$.

The magnetic field $\boldsymbol{B}$ and the gradient along a field line are given by $\boldsymbol{B}=\dot\chi\,\nabla\varphi\times\nabla\upsilon$ and $\boldsymbol{B}\cdot\nabla=\dot\chi\,\partial/\partial\vartheta$, respectively. Setting

$$\boldsymbol{\xi} = U\,\nabla\vartheta\times\nabla\varphi + T\nabla\upsilon\times\nabla\vartheta + S\,\dot\chi\,\nabla\varphi\times\nabla\upsilon, \quad (2)$$

one obtains

$$\boldsymbol{Q}\times\boldsymbol{B} = \dot\chi^2\left[\frac{\partial U}{\partial\vartheta}\nabla\varphi - \frac{\partial T}{\partial\vartheta}\nabla\upsilon\right], \quad (3)$$

$$\boldsymbol{Q}\cdot\boldsymbol{B}-\boldsymbol{\xi}\cdot\nabla p = -B^2\left[2\,(U\varkappa_\upsilon+T\varkappa_\varphi)+\frac{\partial U}{\partial\upsilon} + \frac{\partial T}{\partial\varphi} - \dot\chi\frac{\partial}{\partial\vartheta}\left(\frac{U B_\upsilon+T B_\varphi}{B^2}\right)\right], \quad (4)$$

$$\nabla\cdot\boldsymbol{\xi} = \frac{\partial U}{\partial\upsilon} + \frac{\partial T}{\partial\varphi} + \dot\chi\frac{\partial S}{\partial\vartheta}, \quad (5)$$

$$\boldsymbol{B}\times\boldsymbol{\xi}^*\cdot\boldsymbol{Q} = \dot\chi^2\left[T^*\frac{\partial U}{\partial\vartheta} - U^*\frac{\partial T}{\partial\vartheta}\right], \quad (6)$$

$$(\boldsymbol{\xi}\cdot\nabla p)(\boldsymbol{\xi}^*\cdot\boldsymbol{\varkappa}) = \dot p\,[\varkappa_\upsilon|U|^2+\varkappa_\varphi\,U T^*], \quad (7)$$

where B_υ, B_φ, $\varkappa_\upsilon$, $\varkappa_\varphi$ are covariant of $\boldsymbol{B}$ and $\boldsymbol{\varkappa}$.

The perturbations $\boldsymbol{\xi}$ considered here are localized around a field line defined by $\upsilon=\upsilon_0$, $\varphi=\varphi_0$. Setting

$$t=(\upsilon-\upsilon_0)/\upsilon_0\,\varepsilon, \quad x=(\varphi-\varphi_0)/\varepsilon^2, \quad (8)$$

with $\varepsilon\ll 1$, we require that $\partial_t\sim\partial_x\sim O(1)$, when applied to perturbations. Furthermore, the perturbations are required to vanish for $|\upsilon-\upsilon_0|\geqq\upsilon_0\,\varepsilon$, $|\varphi-\varphi_0|\geqq\varepsilon^2$, i.e.

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$|t|$, $|x| \geqq 1$. Notice that, contrary to the case with shear, the perturbations are less localized in the radial direction.

The equilibrium quantities $A(v, \vartheta, \varphi)$ are now expanded in a Taylor series around $v = v_0$, $\varphi = \varphi_0$:

$$A(v, \vartheta, \varphi) = A(v_0, \vartheta, \varphi_0) + \frac{\partial A}{\partial v}(\vartheta_0, \vartheta, \varphi_0)(v - v_0) + O(\varepsilon^2),$$

and the test functions U, T, S in series of the form

$$U = U_0 + U_1 \varepsilon + U_2 \varepsilon^2 + \ldots,$$

where the order of magnitude of the different terms is given by the powers of ε, and the functions U_i may depend implicitly on ε (similarly for T and S). Then, from the lowest order δW, the following conditions are obtained: $T_0 = 0$ and $\partial_t U_0 + v_0 \partial_x T_1 = 0$ (otherwise, δW will be positive).

Then, in this approximation, one has

$$\boldsymbol{Q} \times \boldsymbol{B} = \dot{\chi}^2 \frac{\partial U_0}{\partial \vartheta} \nabla \varphi + O(\varepsilon), \tag{9}$$

$$\boldsymbol{Q} \cdot \boldsymbol{B} - \xi \cdot \nabla p = -B^2 \left[2 U_0 \varkappa_v + \frac{1}{v_0} \frac{\partial U_1}{\partial t} + \frac{\partial T_2}{\partial x} - \dot{\chi} \frac{\partial}{\partial \vartheta} \left(\frac{U_0 B_v}{B^2} \right) \right] + O(\varepsilon), \tag{10}$$

$$\nabla \cdot \xi = \frac{1}{v_0} \frac{\partial U_1}{\partial t} + \frac{\partial T_2}{\partial \chi} + \dot{\chi} \frac{\partial S_0}{\partial \vartheta} + O(\varepsilon), \tag{11}$$

$$\boldsymbol{B} \times \xi^* \cdot \boldsymbol{Q} = O(\varepsilon), \tag{12}$$

$$(\xi \cdot \nabla p)(\xi^* \cdot \boldsymbol{\varkappa}) = \dot{p} \varkappa_v |U_0|^2 + O(\varepsilon). \tag{13}$$

The function S_0 only appears in the term $\nabla \cdot \xi$ and can be used in the minimalization process to eliminate the contribution of the oscillating part (oscillating in ϑ!) of $h := \partial U_1 / v_0 \partial t + \partial T_2 / \partial \chi$. This is different in the case with shear, where the term $\nabla \cdot \xi$ can be completely eliminated. Then, subsequent minimalization of δW with respect to h leads, in lowest order in ε, to the following expression for the potential energy:

$$\delta W_0(v_0, \varphi_0) = \frac{v_0 \varepsilon^3}{2} \int_{-1}^{1} \mathrm{d}t \int_{-1}^{1} \mathrm{d}x \int_{0}^{N} \mathrm{d}\vartheta \left[\frac{\dot{\chi}^4}{B^2} |\nabla \varphi|^2 \left| \frac{\partial U_0}{\partial \vartheta} \right|^2 + \frac{4 \gamma_H p}{\left\langle 1 + \frac{\gamma_H p}{B^2} \right\rangle} |\langle \varkappa_v U_0 \rangle|^2 - 2 \dot{p} \varkappa_v |U_0|^2 \right]. \tag{14}$$

Here, the equilibrium quantities only depend on ϑ, the variable along the field line (v_0 and φ_0 enter the calculation only as parameters), and $\langle \ldots \rangle$ denotes the mean value on a field line. The first stabilizing term in δW_0 represents the work done against the tension of the magnetic field lines, while the second term represents the work necessary to compress the fluid and does not appear in a similar minimalization of ∂W in equilibria with shear. The non-positive-definite term arises from the interaction of the perturbation with the magnetic line curvature and is the only destabilizing factor here.

Finally, minimizing δW_0 with respect to U_0, one obtains (with a Lagrange multiplier Λ) the Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta} \left[\frac{\dot{\chi}^4}{B^2} |\nabla \varphi|^2 \frac{\mathrm{d}F}{\mathrm{d}\vartheta} \right] + 2 \dot{p} \varkappa_v F + \Lambda F = \frac{4 \gamma_H p}{\left\langle 1 + \frac{\gamma_H p}{B^2} \right\rangle} \varkappa_v \langle \varkappa_v F \rangle, \tag{15}$$

where $U_0 = u(x, t) F(\vartheta)$, $F(\vartheta) = F(\vartheta + N)$. It thus follows that

$$\delta W_0(v_0, \varphi_0) = \frac{v_0 \varepsilon^3}{2} \Lambda \left\{ \int_{-1}^{1} \mathrm{d}x \int_{-1}^{1} \mathrm{d}t |u|^2 \right\} \int_{0}^{N} |F|^2 \mathrm{d}\vartheta. \tag{16}$$

Hence, one has instability if $\Lambda_{\min} < 0$, where $\Lambda_{\min}$ is the lowest eigenvalue of (15).

Equation (15) differs from that obtained in the low shear limit of the ballooning mode criterion for equilibria with shear [1] through the stabilizing term on the right-hand side. The eigenvalues of (15) depend on the field line through the parameters v_0, φ_0 and also on the choice of the coordinates: $\nabla \varphi$ and the covariant basis vector $\nabla \vartheta \times \nabla \varphi$ are arbitrary in the sense that the coordinate φ is determined by the equilibrium only within an arbitrary function of the volume v. One thus obtains different criteria for different choices of this function. Methods discussed in [4], page 37, can be used to relate the sign of the lowest eigenvalue Λ_1 of (15) to the sign of the two lowest eigenvalues of the homogeneous equation obtained by omitting the right-hand side of (15).

The class of criteria obtained here is related to the stability condition for the continuous spectra which correspond to singularities not in the direction of the pressure gradient [5, 6].

The criteria given by (15) can also be obtained from the linearized equations of resistive MHD. To do so in the simplest way, we consider the previously derived equations governing resistive localized modes in *sheared* equilibria. These are Eqs. (13)–(19) of references [2, 3]. In these equations, the shear $\dot{q}$ is now set to 0, and it is assumed that the radial gradients of the perturbations are much smaller than the x-derivatives. (Notice that t and x are defined in [2, 3] in a slightly different way). The appropriate ansatz to solve these equations is then

$$\Phi(t, \vartheta, x) = f(\varepsilon t, \varepsilon^2 x)\, e^{i\alpha x} F(\vartheta), \tag{17}$$

$$(\delta \boldsymbol{B} \cdot \boldsymbol{B})_0 (t, \vartheta, x) = f(\varepsilon t, \varepsilon^2 x)\, e^{i\alpha x} b(\vartheta), \tag{18}$$

$$i\alpha \dot{p} v_0 D(\vartheta) := b(\vartheta) - i\alpha \dot{p} v_0 F(\vartheta), \tag{19}$$

and the stability problem of resistive localized modes in equilibria *with closed field lines* can be reduced to solving two coupled differential equations for $F(\vartheta)$ and $D(\vartheta)$, namely

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left[\frac{|\nabla\varphi|^2}{B^2(1+(\alpha^2\eta^*/\gamma)|\nabla\varphi|^2)}\frac{\mathrm{d}F}{\mathrm{d}\vartheta}\right] - \frac{\varrho\gamma^2}{\dot{\chi}^2 B^2}|\nabla\varphi|^2 F = -\frac{2\dot{p}}{\dot{\chi}^4}\varkappa_v (F+D), \tag{20}$$

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left[\frac{1}{B^2}\frac{\mathrm{d}D}{\mathrm{d}\vartheta}\right] - \frac{\varrho\gamma^2}{\dot{\chi}^2}\frac{(\gamma_H p + B^2)}{\gamma_H p B^2} D - \frac{\varrho\alpha^2}{\dot{\chi}^2 B^2}\eta^*\gamma|\nabla\varphi|^2 D - \frac{2\dot{p}}{\dot{\chi}^4}\frac{\alpha^2\eta^*}{\gamma}\varkappa_v (D+F) = \frac{2\varrho\gamma^2}{\dot{p}\dot{\chi}^2}\varkappa_v F. \tag{21}$$

These equations are formally the same as those describing resistive ballooning modes in equilibria *with shear*, with $\dot{q}$ set to 0. The crucial difference is that here the equations must be solved *with different boundary conditions, namely* $F(\vartheta+N) = F(\vartheta)$, $D(\vartheta+N) = D(\vartheta)$, and not in the infinite interval.

Setting $\eta^* = 0$ yields

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left[\frac{|\nabla\varphi|^2}{B^2}\frac{\mathrm{d}F}{\mathrm{d}\vartheta}\right] - \frac{\varrho\gamma^2}{\dot{\chi}^2 B^2}|\nabla\varphi|^2 F = -\frac{2\dot{p}}{\dot{\chi}^4}\varkappa_v (F+D), \tag{22}$$

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left[\frac{1}{B^2}\frac{\mathrm{d}D}{\mathrm{d}\vartheta}\right] - \frac{\varrho\gamma^2}{\dot{\chi}^2}\frac{(\gamma_H p + B^2)}{\gamma_H p B^2} D = \frac{2\varrho\gamma^2}{\dot{p}\dot{\chi}^2}\varkappa_v F, \tag{23}$$

which are the appropriate equations to describe ideal ballooning modes in equilibria with closed field lines.

Near marginal stability, the growth rate γ is small (γ^{-1} is large compared with the time needed for sound and Alfvén wave propagation). In this limit, and introducing $\hat{D}$ by

$$D := \hat{D} - \frac{2}{\dot{p}}\frac{\gamma_H p}{\left\langle 1 + \frac{\gamma_H p}{B^2}\right\rangle}\langle \varkappa_v F\rangle, \tag{24}$$

one can obtain from (22) and (23) the relation

$$\gamma^2 \int_0^N |r|^2 \,\mathrm{d}\vartheta = \int_0^N \left[-\frac{|\nabla\varphi|^2}{B^2}\left|\frac{\mathrm{d}F}{\mathrm{d}\vartheta}\right|^2 + \frac{2\dot{p}}{\dot{\chi}^4}\varkappa_v F^2 - \frac{4}{\dot{\chi}^4}\frac{\gamma_H p}{\left\langle 1+\frac{\gamma_H p}{B^2}\right\rangle}\langle\varkappa_v F\rangle \varkappa_v F\right]\mathrm{d}\vartheta, \tag{25}$$

where $|r|^2$ is a positive definite function of F and $\mathrm{d}\hat{D}/\mathrm{d}\vartheta$. It is clear that the sign of γ^2 and that of $-\Lambda$ are equal, and (25) is equivalent to (15).

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